SOLUTIONS:

Exercise 1

(*i*) $\sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33}$.

(*ii*) $A_{ij}B_{ij} = A_{11}B_{11} + A_{12}B_{12} + A_{13}B_{13} + A_{21}B_{21} + A_{22}B_{22} + A_{23}B_{23} + A_{31}B_{31} + A_{32}B_{32} + A_{33}B_{33}$.

(*iii*) For i=1 et j=1, $\sigma_{11}=\lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})+2\mu\varepsilon_{11}$.

For $i = 1$ et $j = 2$, $\sigma_{12}=2\mu\varepsilon_{11}$.

In the same manner we can write the other terms for (i,j) = (2,2) , (3,3) , (1,3) , (3,1) , (2,1) , (2,3) , (3,2) .

(*iv*) $$\frac{\partial^2 F}{\partial x_i \partial x_i} = \frac{\partial^2 F}{\partial x_1 \partial x_1} + \frac{\partial^2 F}{\partial x_2 \partial x_2} + \frac{\partial^2 F}{\partial x_3 \partial x_3}.$$

(*v*) Note that,

$$\frac{\partial \sigma_{i1}}{\partial x_1} + \frac{\partial \sigma_{i2}}{\partial x_2} + \frac{\partial \sigma_{i3}}{\partial x_3} + b_i = 0.$$

Thus, for $i = 1$ we have,

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + b_1 = 0.$$

Similar expressions are obtained for i = 2, 3.

Exercise 2

(*i*) $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$

(*ii*)
$$\begin{aligned}\delta_{ij}\delta_{ij} &= \delta_{11}\delta_{11} + \delta_{12}\delta_{12} + \delta_{13}\delta_{13} \\ &\quad + \delta_{21}\delta_{21} + \delta_{22}\delta_{22} + \delta_{23}\delta_{23} \\ &\quad + \delta_{31}\delta_{31} + \delta_{32}\delta_{32} + \delta_{33}\delta_{33} \\ &= \delta_{ii} = \delta_{jj} \\ &= \delta_{11}\delta_{11} + \delta_{22}\delta_{22} + \delta_{33}\delta_{33} \\ &= 3.\end{aligned}$$

(*iii*) $\delta_{ij}A_{ik} = \delta_{1j}A_{1k} + \delta_{2j}A_{2k} + \delta_{3j}A_{3k} = A_{jk}$.

Exercise 3:

Given that the indices in every term are all dummy indices and that the order of x_i is not important, all terms are equivalent. We can write successively,

$$
\begin{aligned}
&(P_{ijk} + P_{jik} + P_{ikj})x_i x_j x_k \\
&= P_{ijk}x_i x_j x_k + P_{jik}x_i x_j x_k + P_{ikj}x_i x_j x_k \\
&= P_{ijk}x_i x_j x_k + P_{mnp}x_n x_m x_p + P_{qrs}x_q x_s x_r \\
&= P_{ijk}x_i x_j x_k + P_{mnp}x_m x_n x_p + P_{qrs}x_q x_r x_s \\
&= 3P_{ijk}x_i x_j x_k .
\end{aligned}
$$

Exercise 4:

Use the divergence, or Gauss theorem: $\int_{\Omega} \frac{\partial T_{jk..}}{\partial x_i} dV = \int_{\partial\Omega} n_i T_{jk..} ds$ with $T_{jk..} = x_i$ we can write,

$$\int_{\partial\Omega} x_i n_j ds = \int_{\Omega} \frac{\partial x_i}{\partial x_j} dV = \delta_{ij} \int_{\Omega} dV = \delta_{ij} V .$$

Exercise 5:

Use the divergence theorem: $\int_{\Omega} \frac{\partial T_{jk..}}{\partial x_i} dV = \int_{\partial\Omega} n_i T_{jk..} ds$.

Note that $\boldsymbol{b} = \nabla \times \boldsymbol{u}$ in component form is $b_i = \varepsilon_{ijk} u_{k,j}$. Thus, with $T_{jk..} = \lambda b_i = \lambda \varepsilon_{ijk} u_{k,j}$ we have

$$\int_{\partial\Omega} n_i \left(\lambda \varepsilon_{ijk} u_{k,j}\right) dS = \int_{\Omega} \frac{\partial\left(\lambda \varepsilon_{ijk} u_{k,j}\right)}{\partial x_i} dV = \int_{\Omega} \varepsilon_{ijk} \partial\left(\lambda u_{k,j}\right)_{,i} dV = \int_{\Omega} \left(\varepsilon_{ijk} \lambda_{,i} u_{k,j} + \varepsilon_{ijk} \lambda u_{k,ji}\right) dV .$$

Because $u_{k,ji}$ is symmetric, the second term in the integrand is zeo. Thus, we have,

$$\int_{\partial\Omega} n_i \left(\lambda \varepsilon_{ijk} u_{k,j}\right) dS = \int_{\Omega} \left(\lambda_{,i} b_i\right) dV .$$

Exercise 6: Using index notation we can write,

(a).

$$\boldsymbol{u} \cdot \boldsymbol{L}^T \boldsymbol{v} = u_i \left(\boldsymbol{L}^T\right)_{im} v_m = u_i L_{mi} v_m = L_{mi} u_i v_m = (\boldsymbol{L}\boldsymbol{u} \cdot \boldsymbol{v})$$

$$\boldsymbol{u} \cdot \boldsymbol{L}^T \boldsymbol{v} = u_i \left(\boldsymbol{L}^T\right)_{im} v_m = u_i L_{mi} v_m = v_m L_{mi} u_i = (\boldsymbol{v} \cdot \boldsymbol{L}\boldsymbol{u})$$

(b) $\left[(\boldsymbol{u} \otimes \boldsymbol{v})(\boldsymbol{a} \otimes \boldsymbol{b})\right]_{ij} = u_i v_m a_m b_j = v_m a_m u_i b_j = \left[(\boldsymbol{v} \square \boldsymbol{a})(\boldsymbol{u} \otimes \boldsymbol{b})\right]_{ij}$

or, $(\boldsymbol{u} \otimes \boldsymbol{v})(\boldsymbol{a} \otimes \boldsymbol{b}) = (\boldsymbol{v} \square \boldsymbol{a})(\boldsymbol{u} \otimes \boldsymbol{b})$.

Note that $(\boldsymbol{v} \square \boldsymbol{a})$ is a scalar and can be placed before or after the dyadic product.

Exercise 7:

In a general manner we can write,

$$\lambda = A_{ij}L_{ij} = \frac{1}{2}(A_{ij}L_{ij} + A_{ij}L_{ij}) = \frac{1}{2}(A_{ij}L_{ij} + A_{ji}L_{ji})$$

Note that since the terms is a scalar we can swap the indices in the second term. Using the symmetry of A_{ij}, we can write,

$$\lambda = \frac{1}{2}(A_{ij}L_{ij} + A_{ij}L_{ji}) = \frac{1}{2}A_{ij}(L_{ij} + L_{ji})$$

By definition we have, $L^s_{ij} = (L_{ij} + L_{ji})/2$ and thus, $A_{ij}L_{ij} = A_{ij}L^s_{ij}$.

Exercise 8:

Define $A_{ij} = x_i x_j$. Then A_{ij} are the components of the symmetric tensor,

$$\boldsymbol{A} = \boldsymbol{x} \otimes \boldsymbol{x},$$

Using the conclusion of the previous exercise, we see that the quantratic form,

$$\boldsymbol{x}\cdot\boldsymbol{L}\boldsymbol{x} = L_{ij}x_i x_j = \frac{1}{2}(L_{ij}x_i x_j + L_{ji}x_j x_i) = \frac{1}{2}x_i x_j(L_{ij} + L_{ji}) = x_i x_j L^s_{ij},$$

does not change if $\boldsymbol{L}$ is replaced by $\boldsymbol{L}^s$.

Execise 9:

$$\frac{\partial S}{\partial x_k} = A_{ij}\frac{\partial x_i}{\partial x_k}x_j + A_{ij}x_i\frac{\partial x_j}{\partial x_k} = A_{ij}\delta_{ik}x_j + A_{ij}x_i\delta_{jk} = A_{kj}x_j + A_{ik}x_i = A_{kj}x_j + A_{jk}x_j = \left(A_{kj} + A_{jk}\right)x_j$$

$$\Rightarrow \frac{\partial S}{\partial x_i} = \left(A_{ij} + A_{ji}\right)x_j.$$

$$(A_{ij} + A_{ji})\frac{\partial x_j}{\partial x_k} = \left(A_{ij} + A_{ji}\right)\delta_{jk} = A_{ik} + A_{ki} = A_{ij} + A_{ji}.$$

When $A_{ij} = A_{ji}$. With the symmetry we have, $2A_{ij}x_j$ and $2A_{ij}$.